The Iterative Conception and its Justification of Large Cardinals

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Abstract

The aim of this dissertation is to discuss the support for inaccessible and Mahlo cardinals based on reflection principles and the iterative conception. Including reflection in the iterative conception allows us to build a hierarchy of stronger reflection principles. The dissertation will exhibit derivations of the existence of inaccessible and Mahlo cardinals from these principles and critically discuss the use of second-order logic involved.

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Chapter 1

Introduction

Large cardinals – cardinals that cannot be obtained by any of the methods provided by ZFC – are an active area of current research, with applications in descriptive set theory, combinatorics and consistency strength of axioms. Since their existence does not follow from ZFC, and in fact even proves its consistency, it is important to justify our use of large cardinals. In this dissertation, we will focus on *inaccessible* and *Mahlo cardinals*, two "small" kinds of large cardinal which both give rise to models of ZFC.

We begin by setting out and justifying an elegant axiomatisation of the iterative conception of set theory due to Dana Scott (1974). The iterative conception is a natural and popular view of set theory which stratifies the universe of sets into stages.

In particular, Scott's axiomatisation contains a *reflection* axiom. In the second part of the dissertation, we show how reflection can be strengthened to give models of ZF, following Lévy (1960). We critically discuss Lévy's argument that it is a natural step to move from this strengthened reflection to a principle that gives the existence of inaccessible cardinals, concluding that his assumption of second-order logic is in fact too strong a premise.

In the last two sections, we prove that if we assume this stronger principle, the inaccessible cardinals are in fact unbounded. Further, we build a hierarchy of increasingly powerful reflection principles which allow us to deduce the existence of Mahlo cardinals. The intention of these two results is to motivate further research on the justification of the "missing step" in Lévy's argument from Scott's reflection axiom to Lévy's stronger axioms, since they have such desirable consequences. Further applications of reflection principles to large cardinals are mentioned in the conclusion.

This dissertation is aimed at any mathematician or logician of fourth-year undergraduate level or beyond. Familiarity with the axioms of ZFC will be assumed, as well as some knowledge of model theory.

Chapter 2

The Iterative Conception

The idea that we form sets by taking elements and 'gathering' them into a set is widespread and dates back to Cantor's much-quoted notion that a set is

> "any collection . . . into a whole of definite, well-distinguished objects . . . of our intuition or thought" (as translated in Boolos, 1971)

We naturally think: here are some things, now we gather them into a collection (a "whole"). *Now* we have a set. So in some sense, the elements of a set exist *prior* to it.

More concretely, if we assume there are no individuals (non-sets)[1] then at first the only collection, i.e. set we can form is empty. At the next stage, we form all collections of objects we had previously. We keep going, reaching an ω'th stage and continuing beyond it, in fact "keeping going" beyond any ordinal we may come across. Note that at each stage we end up repeating the sets we had at previous stages. This process stratifies our universe of sets into stages to which the sets belong and is called the *iterative conception* or *iterative hierarchy*. A key idea is that we always collect *all* possible sets at each stage, and continue (transfinitely) as far as possible. (This is sometimes known as the maximal iterative conception (Wang, 1983)).

One might ask how can we use ordinals to enumerate our stages, if we are trying to build set theory from scratch. We will see when we axiomatise the iterative conception that there is a neat way of doing without ordinals – and even without Foundation – and then deriving these notions from our axioms.

This difficulty aside, this conception of set formation and set theory seems a natural and coherent one. Boolos (1989) (as interpreted by Paseau, 2007) argues that it is a unified picture – no parts of it seem as easy to justify on their own as within the conception – and that the principles within it are very simple, in particular more simple than their consequences. Further it is a familiar and actual conception used by many mathematicians, so we will henceforth assume it is a solid foundation on which to build the rest of our justification and mathematics.[2]

[1]While the iterative conception may often include individuals or *urelements*, hereditary sets are sufficient to express any set we may wish to consider, so we need not assume any individuals.

[2]For excellent and more detailed justifications of the iterative conception, see Boolos (1971) or Wang (1983).

2.1 Axiomatisation

Why use a different axiomatisation for the iterative conception to ZFC? It is, as we will see in Section 2.3.2, perfectly possible to express the iterative hierarchy of sets within ZFC. However, this fails to capture the idea that the iterative conception is at the base of our understanding of sets.

There are many axiomatisations of the iterative conception. A well-known one is given in Boolos (1971). We will use Dana Scott's 1974 axiomatisation. Its axioms have the virtues of being compact and, except possibly in the case of Reflection, very intuitive. Reflection is a concise and powerful axiom that will be very useful to us later in linking the iterative conception and large cardinals.

Our first two axioms are shared with ZFC.

Extensionality. $\forall x \, \forall y \, (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$.

Comprehension. $\forall x \, \exists y \, \forall z \, (z \in y \leftrightarrow (z \in x \wedge \varphi(z)))$ (where φ may have other free variables).

The remaining axioms relate specifically to the iterative conception. We assume for now that we can quantify over stages V, V', V'', etc. Within ZFC, it is possible to remove this by quantifying over ordinals – sets with a certain property – and identifying each stage with an ordinal.

Recall that given a stage, we form the next stage by taking all the sets of sets in the given stage, which includes any sets we had before. Hence a "later" stage contains all sets and subsets of "earlier" stages, and further contains nothing else. We take our stages to be sets in their own right, namely sets of their members, and denote 'V' is earlier than V' simply by $V' \in V$.

Accumulation. $\forall V' \, \forall x \, (x \in V' \leftrightarrow (\forall y \; (y \notin x) \vee \exists V \in V'(x \in V \vee x \subseteq V)))$.

We will see later that there is a first stage, which then must contain nothing but the empty set.[3]

We next express the idea that the iterative hierarchy of stages captures all sets eventually.

Restriction. $\forall x \, \exists V (x \subseteq V)$.

This axiom gives the existence of at least one level. Our final axiom tells us more about the "extent" of our stages. We use φ^V to denote the *relativisation* of φ to V.

Reflection. $\exists V \; \forall x \in V(\varphi(x) \rightarrow \varphi^V(x))$. φ may contain other free variables beyond x. (Note that the given V depends on φ.)

That is, for each $\varphi(x)$ which is true in the whole universe, we can find some subset of the universe in which φ also holds. For example, if for all sets x in the universe there exists $y = \{x\}$ in the universe, then there exists V such that for all $x \in V$, there is $y = \{x\} \in V$. So applying 'set of' never takes us outside of V. Further, our stages get very large: if we have some ordinal[4] α, then we can take $\varphi \equiv \exists x \; (x = \alpha)$ to get $\exists V \; \alpha \in V$.

[3] Scott's version of Accumulation includes urelements which occur in all stages. As we assume there are no non-sets, the relevant clause becomes 'the empty set is contained in all stages'.

[4] By Section 3.3, the notion of ordinals does not change when relativised to V.

While the other axioms seemed very intuitive, Reflection is not quite so obvious. We will discuss below why we should accept Reflection, and show in Section 2.3 that Reflection is equivalent to the ZF axioms Infinity and Replacement.

2.2 Choosing axioms

Maddy (1988) argues that the axioms of ZFC need justifying rather than just stating at the beginning of every undergraduate textbook. In the same spirit we should discuss why we chose precisely these five axioms.

Extensionality is viewed by many, following Boolos (1971, pp. 229-230), to be *analytic*, i.e. expressing, or contained in, or equivalent to the definition of set.[5] Hence our axiomatisation must contain Extensionality in order to be an axiomatisation of *set* theory. For if someone says "there are two distinct sets whose members are the same", we are far more likely to conclude that they do not mean what they say than if they contradict any other axiom. (For example, that they do not mean 'set', but rather some collection with additional defining properties.)

The same can be said to a lesser degree about Comprehension. Given a set x, the ability to take a subset of it satisfying a certain property is certainly part of our intuition about sets. (Arguably, we intuitively think we ought to be able to take any set with a certain property, but of course this leads to Russell's Paradox. Comprehension may not fully capture our intuition, but it is certainly a part of it.)

Accumulation and Restriction are central to the iterative conception – in a sense, they are also analytic for it. Accumulation is essentially a definition of stages, while Restriction states that the iterative conception "works", i.e. captures everything.

The two also gain some support from a principle Maddy calls *Maximize*, i.e. that our set theory should be as "broad" and as "tall" as possible, i.e. should include as many sets, as possible. The ordinals should be inexhaustible (whatever operation we apply, we can always find an ordinal beyond it) and power sets should be very rich or "thick". In this sense, Accumulation gives the breadth of the hierarchy and the richness of power sets (as it collects all possible subsets into the next stage) while Restriction gives the height of the hierarchy.

The beauty of this axiomatisation when compared to ZFC is that no arbitrariness appears in our choice of axioms so far (Scott, 1974, p. 212). Assuming we believe in the iterative conception, each axiom follows self-evidently from the conception (and the notion of set). ZFC is certainly a practical axiomatisation but it is slightly harder to see why – beyond those practical reasons – we should pick some of its axioms over others.

Our Axiom of Reflection is a (weak) formulation of a broader principle of Reflection. The principle states that the universe is so rich and so complex that any description we may have of it in fact is also true of just a small part of it, usually a set or proper class. In other words, any attempt at "the universe is the unique collection satisfying …" fails, as this property will *reflect* downwards to some subset.

[5] This is, as Boolos himself says, under the assumption that any distinction can be made between analytic and synthetic statements at all.

Reflection is on the whole very popular. Maddy describes it as “the most universally accepted rule of thumb in higher set theory” (1988, p. 503). Notable proponents are Gödel (1989, p. 146), Reinhardt (1974) and Kanamori and Magidor (1977).

We can prove weak reflection principles within ZFC (see Section 2.3.2), thus lending further support to their validity (if we assume ZFC is true). We do not claim to capture reflection fully in our axiom. We will later see stronger axioms which encapsulate more of it, but must also fall short by the principle's very nature. (If we could find a reflection principle which described the whole universe, there must be a proper class of the universe in which it holds ...)

The intuitiveness of Reflection is contested. Paseau (2007, p. 44) asserts that Reflection is highly intuitive, while Scott (1974) asserts that it is not as intuitive as Infinity and Replacement, its ZFC counterparts (see Section 2.3). Certainly to the novice set theorist, Infinity and Replacement may be easier to comprehend. However Maddy (2011) and others have argued that it takes experience of working with set theory to fully appreciate the intuitiveness of an axiom. Finally, Scott argues that while Reflection may not be as intuitive, it has “all the practical advantages of a good axiom” (1974, p. 213) (for example, it is powerful, simple to state, and unifies existing principles (Wang, 1983, p. 551)).

2.3 Equivalence to ZF

2.3.1 Derivation of ZF from the iterative conception

To satisfy ourselves that the set theory we have axiomatised is our usual one, we will proceed to derive the axioms of ZF, largely following Scott (1974).

It immediately follows from Restriction that a stage exists, and hence by Accumulation it contains the empty set, which then exists in our stage theory.

Theorem 2.1 (Empty Set). $\exists V \exists x \in V \forall y\ \neg y \in x$.

We next show the Axiom of Foundation. Scott's axiomatisation stands out in not needing to assume Foundation (or equivalently a well-founded class of ordinals corresponding to the stages).

Theorem 2.2 (Transitivity of stages). $\forall V, V'\ V \in V' \rightarrow V \subseteq V'$.

Proof. Suppose $x \in V \in V'$. By the reverse implication of Accumulation we immediately get $x \in V'$. □

Definition 2.1. We call a set x *grounded* if $\forall a(x \in a \rightarrow \exists y \in a\ \forall z \in a\ z \notin y)$. That is, if and only if x is contained only in well-founded sets. Further define for every stage V

$$\|V\| = \{x \in V : \forall a(x \in a \rightarrow \exists y \in a\ y \cap a = \varnothing)\}$$

to be the set of grounded subsets of V (where '$y \cap a = \varnothing$' abbreviates $\forall z \in a\ z \notin y$).

We will show that in fact all stages are grounded and hence derive Foundation.

Lemma 2.3 (Grounded subsets of stages). $\forall V (V \in V' \rightarrow \|V\| \in \|V'\|)$

Proof. Suppose $V \in V'$. By Accumulation $\|V\| \in V'$. Suppose $\|V\| \in a$ for some a. If $\|V\| \cap a \neq \varnothing$, then by definition of $\|V\|$ there is $y \in a$ such that $y \cap a = \varnothing$. Hence $\|V\|$ is well-founded in V' and so $\|V\| \in \|V'\|$. □

Theorem 2.4 (Transitivity)**.** $\forall V\, \forall x\ (x \in V \to x \subseteq V)$.

Proof. Suppose $x \in V$, set $a = \{\|V'\| : V' \in V \wedge x \in V'\}$. By Comprehension over $\|V\|$, the set $a \subset \|V\|$ is well-defined. Suppose $a = \varnothing$, then $\forall V' \in V\ x \notin V'$. Accumulation gives $x \subseteq V' \in V$ so by transitivity of stages $x \subseteq V$.
Suppose $a \neq \varnothing$, then as $a \subseteq \|V\|$ find $\|V'\| \in a$ such that $\|V'\| \cap a = \varnothing$. As $x \in V'$, Accumulation gives $\exists V'' \in V'(x \in V'' \vee x \subseteq V'')$. The case $x \in V''$ is impossible as then $\|V''\| \in \|V'\| = y$ (by Lemma 2.3) and $\|V''\| \in a$ by definition of a, contradicting $a \cap \|V'\| = \varnothing$. So $x \subseteq V'' \subseteq V' \subseteq V$ as required. □

Theorem 2.5 ((Full) Foundation)**.** *Let φ be any formula. Then*

$$(\exists x\ \varphi(x) \to \exists x(\varphi(x) \wedge \forall y \in x\ \neg\varphi(y)))$$

Proof. Assume $\varphi(z)$. By Restriction, $z \subseteq V$ for some V. Again let

$$a = \{\|V'\| : V' \in V \wedge \exists x \subseteq V'\ \varphi(x)\}$$

By Lemma 2.3 $a \subseteq \|V\|$ and so a exists by Comprehension.
Suppose $a = \varnothing$, then there is no V' such that $\exists x \subseteq V'\ \varphi(x)$, equivalently by Transitivity $\exists x \in V'\ \varphi(x)$. Suppose $\exists w \in z\ \varphi(w)$, then $w \in V' \in V$ or $w \subseteq V' \in V$ for some V'' by Accumulation, so we have $\neg\exists w \in z\ \varphi(y)$.
If $a \neq \varnothing$, choose $\|V'\| \in a$ with $\|V'\| \cap a = \varnothing$ as in the previous proof. As $\|V'\| \in a$ there exists $x \subseteq V'$ such that $\varphi(x)$. Suppose $\exists w \in x\ \varphi(w)$, then by Accumulation for w there is $V'' \in V'$ with $w \subseteq V'' \in V'$ or $w \in V'' \in V'$ (so by Transitivity $w \subseteq V'' \in V'$). But then $\|V''\| \in \|V'\| \cap a$, contradicting our choice of V''. □

Corollary 2.6 (Foundation)**.** $\forall a(a \neq \varnothing \to \exists x \in a(\forall y \in a\ y \notin x))$.

We now show that the stages are well-ordered. In particular this will imply a least stage, which is in line with our original notion of "starting" with the empty set, and building sets from there.

Lemma 2.7 (Irreflexivity of stages)**.** $\forall V\ (V \notin V)$.

Proof. Suppose there is V such that $V \in V$. By Comprehension we can form

$$a = \{x \in V : x \notin x\}$$

By construction $a \subseteq V$, so by Accumulation (since $V \in V$), $a \in V$. Proceed as with Russell's paradox (suppose $a \in a \dots$) to get a contradiction. □

Theorem 2.8 (Well-ordering of stages)**.** *The stages $V, V', \dots$ are well-ordered by $\in$.*

Proof. We have already proved irreflexivity and transitivity. Consider a non-empty set of stages given by a property $\psi(V)$. To show it has an *in*-minimal element, set $\varphi(x) \equiv (\psi(x) \wedge \exists V\ x = V)$. Apply full Foundation to φ to get

$$\exists V \psi(V) \to \exists V(\psi(V) \wedge \neg\exists V' \in V \psi(V')) \tag{2.1}$$

It remains to prove linear ordering of the stages:

$$\forall V\, \forall V'(V \in V' \vee V = V' \vee V' \in V)$$

Suppose not, then by (2.1) we can pick V such that $\neg\forall V'(V \in V' \vee V = V' \vee V' \in V)$ and no $V'' \in V$ has this property. By (2.1) again pick V' least so that $V \notin V' \wedge V \neq V' \wedge V' \notin V$. Suppose now $V'' \in V$. Then $V'' \neq V'$ and since $V' \notin V$, by Transitivity $V' \notin V''$ holds. Since V is minimal such that linear ordering does not hold, we must have $V'' \in V'$. Conversely suppose $V'' \in V'$. By choice of V we again have $V'' \neq V$ and $V \notin V''$. By minimality of V' we get $V'' \in V$. So we have shown

$$\forall V''(V'' \in V \leftrightarrow V'' \in V') \tag{2.2}$$

Suppose $x \in V$. By Accumulation there exists $V'' \in V$ such that $x \in V''$ or $x \subseteq V''$. By (2.2) $V'' \in V$ and $x \in V''$ or $x \subseteq V''$, so $x \in V'$ by Accumulation. Similarly if $x \in V'$. So $\forall x(x \in V \leftrightarrow x \in V')$, so $V = V'$, contradiction. □

Theorem 2.9 (Unions)**.** $\forall x \exists y \forall z(z \in y \leftrightarrow \exists w \in x(z \in w))$.

Proof. Fix some x. By Restriction $\exists V(x \subseteq V)$. If $\exists w \in x\ (z \in w)$, then as $w \in x \in V$ by Transitivity $w \in V$ and $z \in V$. So we can apply Comprehension over V to the formula $\varphi(x, z) \equiv \exists w \in x(z \in w)$ to get $\exists y \forall z(z \in y \leftrightarrow (z \in V \wedge \varphi(x, z)))$ (where we can drop $z \in V$). □

From this point onward we also use Reflection. (Scott proves weak versions of Pairs and Power Set without Reflection, but instead we will provide proofs of full Pairs and Power Set which he leaves to the reader.)

Theorem 2.10 (Strong Restriction)**.** $\forall x \exists V x \in V$.

Proof. Let $\varphi(z) \equiv \exists y\ z = y$. Apply Reflection with no free variables, i.e. keeping z free. Then we get $\exists V(\exists y\ z = y \rightarrow \exists y \in V\ z = y)$ which simplifies to $\exists V\ z \in V$. □

Theorem 2.11 (Power Set)**.** $\forall x \exists y \forall z(z \in y \leftrightarrow z \subseteq x)$.

Proof. Fix $x \neq \varnothing$. By Strong Restriction, $\exists V x \in V$. By Accumulation and Transitivity, there is $V' \in V$ such that $x \subseteq V'$. Now let $\varphi(z, x) \equiv z \subseteq x$. By Comprehension, $\exists y \forall z(z \in y \leftrightarrow (z \in V \wedge z \subseteq x))$. The condition $z \in V$ vanishes as $z \subseteq x \subseteq V' \in V$ gives $z \in V$ by Accumulation. □

Theorem 2.12 (Pairs)**.** $\forall x \forall y \exists z \forall w(w \in z \leftrightarrow (w = x \vee w = y))$.

Proof. Fix x, y. As in the proof of Strong Restriction, apply Reflection to

$$\varphi(x, y) \equiv (\exists w\ (x = w) \wedge \exists z\ (y = z))$$

This gives $\exists V\ (x \in V \wedge y \in V)$. Now let $\psi(w) \equiv (w = x \vee w = y)$. By Comprehension over V we get

$$\exists z \forall w(w \in z \leftrightarrow (w \in V \wedge (w = x \vee w = y)))$$

The condition $w \in V$ vanishes as $x, y \in V$.

□

Theorem 2.13 (Infinity)**.** $\exists V(\varnothing \in V \wedge \forall x \in V\ x \cup \{x\} \in V)$. *That is, there exists an inductive stage (and hence an inductive set).*

Proof. Consider $\varphi \equiv (\exists x\ (x = \varnothing) \wedge (\forall x \exists y\ y = x \cup \{x\}))$. By Pairs (Theorem 2.12), for any x the set $\{x\} = \{x, x\}$ exists and by Pairs again, $\{x, \{x\}\}$ exists. So by Unions (Theorem 2.9), $\exists y\ y = x \cup \{x\}$. So φ holds (as $\varnothing \in V$ for all V). Now, by Reflection,

$$\exists V(\varphi \to (\exists x(x \in V \wedge x = \varnothing) \wedge \forall x \in V \exists y(y \in V \wedge y = x \cup \{x\})))$$

φ holds so simplifying the above formula, we get

$$\exists V(\varnothing \in V \wedge \forall x \in V x \cup \{x\} \in V)$$ □

Before we can prove Replacement, we need a "stronger" version of Reflection. (As we can prove it from Reflection alone, it is in fact equivalent.)

Theorem 2.14 (Reflection in n variables)**.**

$$\forall a \exists V(a \in V \wedge \forall y_1, y_2, \ldots y_n \in V(\varphi(y_1, y_2, \ldots, y_n) \leftrightarrow \varphi^V(y_1, y_2, \ldots, y_n)))$$

A full version of Scott's proof sketch in (1974, p. 213) can be found in Appendix A, but as it is long and rather technical we skip it here.

Theorem 2.15 (Replacement)**.** *For each set a and class function φ on a,*

$$\exists V \exists w \in V w = \{y : \exists x \in a\ \varphi(x, y)\}$$

Proof. Suppose $\varphi(u, v; x_1, \ldots, x_n)$ has only the free variables listed. Let

$$\psi(a, x_1, \ldots, x_n) \equiv \forall x \in a \exists y\ \varphi(x, y; x_1, \ldots, x_n)$$

This holds as φ is a class function. Apply strengthened Reflection to a and $\varphi \wedge \psi$ to get

$$\begin{aligned} \exists V(a \in V \wedge & \forall a, x_1, \ldots, x_n, u, v \in V((\varphi(u, v; x_1, \ldots, x_n) \wedge \psi(a, x_1, \ldots, x_n) \\ & \leftrightarrow (\varphi^V(u, v; x_1, \ldots, x_n) \wedge \forall x \in a \exists y \in V \varphi^V(x, y; x_1, \ldots, x_n)))) \end{aligned}$$

(since $x \in a \in V$, then $x \in V$ automatically by Transitivity). Following a very similar argument to Lévy (1960, Theorem 6), we may split this into

$$\forall x_1, \ldots, x_n, u, v \in V(\varphi(u, v; x_1, \ldots, x_n) \leftrightarrow \varphi^V(u, v; x_1, \ldots, x_n)) \tag{2.3}$$

$$\forall x_1, \ldots, x_n \in V(\psi(a, x_1, \ldots, x_n) \leftrightarrow \forall x \in a \exists y \in V \varphi^V(x, y; x_1, \ldots, x_n)) \tag{2.4}$$

Since in (2.4) the arguments x, y of φ are in V, then we may apply (2.3) to get

$$\forall x_1, \ldots, x_n \in V(\forall x \in a \exists y\ \varphi(x, y; x_1, \ldots, x_n) \leftrightarrow \forall x \in a \exists y \in V \varphi(x, y; x_1, \ldots, x_n))$$

Now ψ holds, so we get

$$\forall x_1, \ldots, x_n \in V\ \forall x \in a\ \exists y \in V \varphi(x, y; x_1, \ldots, x_n)$$

By uniqueness of the images of class functions, all y such that $\varphi(x, y)$ for some $x \in a$ are contained in this V. So apply Comprehension on V to get the required set

$$\{y : \exists x \in a \varphi(x, y)\} = \{y \in V : \exists x \in a\ \varphi(x, y)\} \in V$$ □

2.3.2 Derivation of the iterative conception axioms within ZF

For this section we assume ZF instead of Scott's axioms. We build a class model of the iterative hierarchy within the universe of sets specified by ZF and show that it satisfies Scott's axioms. We then show that, assuming Foundation, this model does actually include the whole ZF universe (i.e. Restriction holds).

Definition 2.2 (Cumulative Hierarchy)**.** Recursively define $V : \mathbf{On} \to U$ (where U is the universe):

1. $V(0) = \varnothing$
2. $V(\alpha + 1) = \mathcal{P}(V(\alpha))$
3. $V(\lambda) = \bigcup_{\alpha<\lambda} V(\alpha)$ for λ a limit ordinal

We write V_α for $V(\alpha)$.

This is the counterpart to our previous stages V: we proved that the levels V were well-ordered, so we may denumerate them by ordinals, if they are already defined. For Scott's axiomatisation of stages, we cannot assume ordinals prior to constructing the stages, but having done so we may note that each stage contains as a subset precisely one new transitive set well-ordered by $\in$. We call these sets ordinals and name each stage V by the new ordinal α it contains ($\alpha \subset V$). Hence we may denote it V_α.

Scott's axiomatisation assumes Extensionality and Comprehension for all sets (independent of stages), so we do not need any further derivation.

For the remainder, we need some preliminary properties of the cumulative hierarchy. For proofs, see Kunen (1980, Chapter III.2).

Definition 2.3. The *rank* of a set x is the least α such that $x \in V_\alpha$.

Note that $\text{rank}(x)$ is always a successor ordinal.

Lemma 2.16. *For all $\alpha \in \mathbf{On}$, $V_\alpha \subseteq V_{\alpha+1}$ and $V_\alpha \in V_{\alpha+1}$.*

Theorem 2.17 (Transitivity of stages)**.** $\forall x(x \in V_\gamma \in V_\beta \to x \in V_\beta)$*. In particular, if $V_\delta \in V_\gamma$ then $V_\delta \subseteq V_\gamma$.*

Lemma 2.18. $\forall \alpha, \beta \in \mathbf{On}(\alpha < \beta \to V_\alpha \in V_\beta)$*. Moreover $V_\alpha \subseteq V_\beta$.*

Lemma 2.19 (Transitivity)**.** $\forall x \exists \alpha(x \in V_\alpha \leftrightarrow x \subset \bigcup_{\beta \in \mathbf{On}} V_\beta)$*.*

We may now derive Scott's axioms.

Theorem 2.20 (Accumulation)**.** $\forall V_\alpha \forall x(x \in V_\alpha \leftrightarrow \exists V_\beta \in V_\alpha(x \in V_\beta \vee x \subseteq V_\beta))$.

Proof. Suppose $x \in V_\alpha$, then in particular $\alpha > 0$.
Case 1: $\alpha = \gamma+1$. Then $x \in V_\alpha$ implies $x \subseteq V_\gamma$ or $x \in V_\gamma$ by construction. $V_\gamma \in V_{\gamma+1} = V_\alpha$, so it satisfies the RHS.
Case 2: $\alpha = \lambda$ for some limit λ. Then $\exists \beta < \lambda$ such that $x \in V_\beta$ and by Lemma 2.18 $V_\beta \in V_\lambda$ so V_β satisfies the RHS.

For the reverse, suppose $\exists V_\beta \in V_\alpha \; x \in V$. Then $V_\beta \subseteq V_\alpha$ so $x \in V_\beta$ implies $x \in V_\alpha$. Suppose instead $\exists V_\beta \in V_\alpha \; x \subseteq V_\beta$.
Case 1: $\alpha = \gamma + 1$. If $\beta = \gamma$, $x \subseteq V_\gamma$. If $\beta < \gamma$, then $x \subseteq V_\beta \subseteq V_\gamma$ by Lemma 2.18 so

$x \subseteq V_\gamma$. In either case, as $V_{\gamma+1} = \mathcal{P}(V_\gamma)$ we have $x \in V_{\gamma+1} = V_\alpha$.
Case 2: $V_\alpha = V_\lambda$ for λ a limit ordinal. Let $V = V_\beta$ as before. Then as $\beta < \lambda$, also $\beta + 1 < \lambda$ and $V_{\beta+1} \subseteq V_\lambda$. $x \subseteq V_\beta$ implies $x \in V_{\beta+1}$ as above, so $x \in V_{\beta+1} \subseteq V_\lambda$ gives $x \in V_\lambda = V_\alpha$. □

Lemma 2.21 (Kunen, 1980, Lemma 3.3). *If x is transitive and $\in$ is well-founded on x, then $\exists\alpha\ x \in V_\alpha$.*

Proof. By Transitivity it is enough to show $x \subset \bigcup_{\beta\in\mathbf{On}} V_\beta$. If not, let $y = x \setminus \bigcup V_\beta$, let z be $\in$-minimal in y. If $w \in z$, then $w \notin y$, but $w \in y \in x$ so $w \in x$, so $w \in \bigcup V_\beta$. Thus $z \subset \bigcup V_\beta$ so $z \in \bigcup V_\beta$ by Transitivity, contradicting $z \in x \setminus \bigcup V_\beta$. □

Definition 2.4 (Transitive closure). We define the *transitive closure* $\mathrm{trcl}(x)$ to be the least transitive set containing x. It is explicitly given by $\bigcup\{\bigcup^n x : n \in \omega\}$ where $\bigcup^0(x) = x$ and $\bigcup^{n+1}(x) = \bigcup(\bigcup^n(x))$.

Theorem 2.22 (Restriction). $\forall x \exists V_\beta (x \subseteq V_\beta)$.

Proof. We modify the proof of Theorem 4.1 in Kunen (1980). Fix x and consider $\mathrm{trcl}(x)$. By Foundation, $\in$ is well-founded on $\mathrm{trcl}(x)$, so by Lemma 2.21 $\mathrm{trcl}(x) \in V_\alpha$ for some α. Further $x \subseteq \mathrm{trcl}(x) \in V_\alpha$ so $x \subseteq \mathrm{trcl}(x) \subseteq \bigcup_{\beta\in\mathbf{On}} V_\beta$ and $x \subseteq V_\beta$ for some β. □

In fact we can prove the stronger result $\forall x\ \exists V_\beta(x \in V_\beta)$, either by adapting the proof just given or by following the proof using Reflection in Theorem 2.10. This is an important result – it tells us that the cumulative hierarchy covers the whole universe eventually. Hence, given ZF, the iterative conception "works".

Using the following criterion, we prove a theorem known as *Lévy's Reflection Principle.* In fact, the reflection principle Lévy gives in his original 1960 paper is slightly different (though equivalent). We will use Lévy's original principle (called N) later in this dissertation, but will follow convention in calling this one 'Lévy's Reflection Principle'. This has the Axiom of Reflection as an immediate corollary.

Theorem 2.23 (Tarski-Vaught Criterion). *Let A, B be non-empty transitive classes. Suppose $\varphi_1, \ldots, \varphi_n$ is a subformula-closed list of formulae in the language of set theory, i.e. every subformula of each φ_i also appears in the list. Suppose each φ_i has n_i free variables and we express φ_i without using $\forall$. The following are equivalent:*

1. *For each i, φ_i is* absolute *for A, B. That is,*

$$\forall a_1, \ldots, a_{n_i} \in A\ (\varphi_i(a_1, \ldots, a_{n_i})^A \leftrightarrow \varphi_i(a_1, \ldots, a_{n_i})^B)$$

2. *For every formula $\varphi_i(v_1, \ldots, v_{n_i})$ of the form $\exists t \varphi_j(v_1, \ldots, v_{n_i}, t)$ we have*

$$\forall a_1, \ldots, a_{n_i} \in A\ (\exists t \in B\ \varphi_j(a_1, \ldots, a_{n_i}, t)^B \rightarrow \exists a \in A\ \varphi_j(a_1, \ldots, a_{n_i}, a)^B$$

See Kunen (1980, Lemma 7.3) for a proof.

Theorem 2.24 (Lévy's Reflection Principle). *Suppose $\tilde{W} : \mathbf{On} \to U$ satisfies*

1. *$\alpha < \beta \to W(\alpha) \subseteq W(\beta)$ for all $\alpha, \beta \in \mathbf{On}$*
2. *$W(\lambda) = \bigcup_{\alpha<\lambda} W(\alpha)$ for all limit $\lambda \in \mathbf{On}$*

Then for $W = \bigcup_{\alpha\in\mathbf{On}} W(\alpha)$ *(a class), for all* $\alpha \in \mathbf{On}$ *and for any list of formulae* $\varphi_1, \dots, \varphi_n$*, there exists* $\beta \in \mathbf{On}$ *such that* $\alpha < \beta$ *and*

$$\forall a_1, \dots, a_{n_i} \in W(\beta)\varphi_i^W(a_1, \dots, a_{n_i}) \leftrightarrow \varphi_i^{W(\beta)}(a_1, \dots, a_{n_i})$$

Proof. We follow the proof of Kunen (1980, Theorem 7.4). We try to find some β such that $W(\beta)$ satisfies the second condition of the Tarski-Vaught Criterion. Without loss of generality assume the list $\varphi_1, \dots, \varphi_n$ does not contain $\forall$ and is subformula closed.

For each i such that φ_i is of the form $\exists t\varphi_j(v_1, \dots, v_{n_i}, t)$ define the class function $F_i : W^{n_i} \to \mathbf{On}$ as follows:

$$F_i(w_1, \dots, w_{n_i}) = \begin{cases} 0 & \neg\exists t \in W\varphi_j(w_1, \dots, w_{n_i}, t)^W \\ \eta & \eta \text{ is least such that } \exists t \in W(\eta)\ \varphi_j(w_1, \dots, w_{n_i}, t)^W \end{cases}$$

Now define $G_i : \mathbf{On} \to \mathbf{On}$ by

$$G_i(\gamma) = \max\{\gamma + 1, \sup\{F_i(w_1, \dots, w_{n_i}) : w_k \in W(\gamma)\}\}$$

Since $W(\gamma)$ is a set, by Replacement so is $\{F_i(w_1, \dots, w_{n_i}) : w_k \in W(\gamma)\}$. Hence its supremum exists and G is well-defined.

Define β_n for $n \in \omega$ recursively: $\beta_0 = \max\{\alpha, \omega\}$, $\beta_{n+1} = \max_i G_i(\beta_n)$. Observe that $\alpha \leqslant \beta_n \leqslant \beta_{n+1}$ so $\beta = \sup_{n\in\omega} \beta_n$ is a limit cardinal with $\alpha < \beta$. We check that $W(\beta)$ satisfies the second condition of the Tarski-Vaught Criterion. By construction it is certainly non-empty and transitive. If φ_i is of the form $\exists t\varphi_j(v_1, \dots, v_{n_i}, t)$ then for any $w_1, \dots, w_{n_i} \in W(\beta)$ there is $n \in \omega$ such that $w_1, \dots, w_{n_i} \in W(\beta_n)$ since the β_n are increasing and β is a limit. Suppose $\varphi_i(w_1, \dots, w_{n_i})$ holds, then $F_i(w_1, \dots, w_{n_i}) = \eta > 0$ and there is $t \in W(\eta)$ such that $\varphi_j(w_1, \dots, w_{n_i}, t)$. Moreover $\eta \leqslant G_i(\beta_n) \leqslant \beta_{n+1} \leqslant \beta$, so $t \in W(\beta)$ as required. □

Corollary 2.25 (Reflection in n variables)**.**

$$\forall a\exists V_\beta(a \in V \wedge \forall y_1, y_2, \dots y_n \in V_\beta(\varphi(y_1, y_2, \dots, y_n) \leftrightarrow \varphi^{V_\beta}(y_1, y_2, \dots, y_n)))$$

Proof. Clearly our stages V_α satisfy 1. of Lévy's Reflection Principle by Lemma 2.18 and 2. by definition of V_λ. Further by Restriction $\forall x\exists\alpha\, x \in V_\alpha$. Then φ^W becomes simply φ. Let $\alpha = \text{rank}(a)$, then we get (for $n = 1$, i.e. just one formula φ):

$$\exists\beta \in \mathbf{On}(a \in V_\beta \wedge \forall a_1, \dots, a_n \in V_\beta\varphi(a_1, \dots, a_n) \leftrightarrow \varphi^{V_\beta}(a_1, \dots, a_n)$$ □

Corollary 2.26 (Reflection)**.** $\exists V_\beta\forall x \in V_\beta(\varphi(x) \to \varphi^{V_\beta}(x))$.

This completes our derivation of the iterative hierarchy axioms from ZF, within the model of the V_α, which we have showed covers the whole universe provided we assume Foundation. This leaves us free to use either set of axioms in the rest of the dissertation.

2.4 The Axiom of Choice

The axiomatisation above notably does not include the Axiom of Choice. The reader will no doubt be aware of the long history surrounding the acceptance of the Axiom of Choice, and we do not intend to discuss whether it is a good or "true" axiom.

A more interesting question for this dissertation is whether Choice forms a part of the iterative conception. Essentially, the question is whether choice sets (for a set x, the set b such that $|a \cap b| = 1$ for all $a \in x$) are formed as part of "all possible subsets" in Accumulation. Boolos (1971) argues that this is circular: such a set is formed if and only if Choice holds. Hence while Choice may be true, it holds independently of the iterative conception. Conversely Paseau (2007) presents a "combinatorial" version of Comprehension (and Accumulation) attributed to Bernays: if we can select *any* subset of a set, then it must include the choice set. Along similar lines, Maddy (1988, p. 493) points out that the Axiom of Choice is thought to "thicken" the power set, and is thus supported by Maddy's *Maximize*. Hence it is part of the iterative conception if we take the iterative conception to be maximal (as e.g. Wang (1983) does).

It seems to be a matter of personal opinion which of the two arguments is more convincing. Whether Choice is part of the iterative conception and whether it is "true" or not, we will henceforth assume that it holds. It is indispensable for much of modern mathematics, in particular for the theory of cardinals which we will use heavily in this dissertation. Further, Choice will form a key part in Lévy's argument concerning inaccessible cardinals in Section 3.3. This step will be better justified if Choice benefits from the support of the iterative conception, but will also work if Choice is considered to hold independently.

Chapter 3

Reflection principles and large cardinals

3.1 An alternative form of reflection

In this section we will see that Reflection is not only a natural principle but also very powerful. Lévy (1960, p.234) shows that we may reformulate it as follows. Let S be ZF minus Infinity and Replacement.

Lévy's N_0. $\exists M\, \mathrm{Scm}^{\mathrm{S}}(M) \wedge \forall x_1, \ldots, x_n \in M(\varphi(x_1, \ldots, x_n) \leftrightarrow \varphi^M(x_1, \ldots, x_n))$.

$\mathrm{Scm}^{\mathrm{S}}(M)$ abbreviates that M is a *standard complete model* of S. These were first defined in Shepherdson (1952) under the name *super-complete model*; the naming 'standard complete model' is due to Lévy (1960).[1] We use Kruse (1965, p. 97)'s concise modern definition, except that following Lévy, we will use sets rather than classes.

Definition 3.1. A *standard complete model* of a theory Σ is a transitive set $(M, \in)$ such that Σ^M holds, that is, all the axioms of Σ hold when relativised to M, and $\in$ on M is the standard relation of $\in$ on the universe. We write $\mathrm{Scm}^{\Sigma}(M)$ if this property holds.

Equivalently, if Σ is a set of true sentences (in the universe), a set M is a standard complete model of Σ if and only if Σ is absolute for M.

The assumption '$\mathrm{Scm}^{\mathrm{S}}(M)$' in Lévy's version of Reflection may seem strong, but in fact N_0 is equivalent to our original Reflection axiom in the presence of the rest of ZF. Further we will show that any V_λ for λ a limit satisfies $\mathrm{Scm}^{\mathrm{S}}(V_\lambda)$ (in particular so does V_ω).

Recall that the following notions are Δ_0 and hence absolute (for proofs, see Kunen, 1980, Chapter IV.3).

Lemma 3.1. *Let $A \subseteq B$ be any transitive classes. The expressions '$x = y \cup z$', '$x = y+1$' and 'β is an ordinal' are absolute for A, B.*

[1] Standard complete models are also closely related to Grothendieck universes (Kruse, 1965), which are frequently used in modern mathematics. For example, it has been discussed whether Andrew Wiles' proof of Fermat's Last Theorem depends on them (McLarty, 2010; Conrad et al., 2010).

Theorem 3.2. *Let λ be a limit cardinal. Then V_λ is a standard complete model of S.*

While it is not hard to check the axioms of S as in ZF, we can now make use of our equivalent iterative conception axioms. These follow very easily as we may remain in the language of stages. (For Extensionality and Comprehension, we essentially follow Kunen (1980, pp. 113-114)).

Proof. Extensionality: Suppose $x, y \in V_\lambda$ and $x \neq y$. By Extensionality in the universe there is $t \in x$ such that $t \notin y$ or $t \in y$ such that $t \notin x$. As V_λ is transitive $t \in V_\lambda$ in either case. So $\neg \forall t \in V_\lambda\ (t \in x \leftrightarrow t \in y)$. The converse follows similarly.

Comprehension: We want to show

$$\forall x \in V_\lambda\ \exists y \in V_\lambda\ \forall z \in V_\lambda\ (z \in y \leftrightarrow (z \in x \wedge \varphi(z)^{V_\lambda}))$$

Pick $x \in V_\lambda$ non-empty, let $\alpha = \text{rank}(x)$. Apply Comprehension in the universe to φ^{V_λ} to obtain y. For all $z \in x$ $z \in V_\alpha$ (by Transitivity) so $\forall z \in y$ $z \in V_\alpha$ so by Accumulation for y we get $y \in V_{\alpha+1} \subseteq V_\lambda$.

Accumulation:

$$\forall V_\alpha \forall x (x \in V_\alpha \leftrightarrow (\forall y \in x\ (y \notin x) \vee \exists V_\beta \in V_\alpha (x \in V_\beta \vee x \subseteq V_\beta)))$$

Accumulation is Π_1, so downwards absolute, so also holds in V_λ.

Restriction: We want to show $\forall x \in V_\lambda \exists V_\alpha \in V_\lambda x \subseteq V_\alpha$. Since λ is a limit cardinal, $x \in V_\lambda$ implies $\exists \alpha < \lambda$ $x \in V_\alpha$, so by Transitivity trivially $x \subseteq V_\alpha$. □

Theorem 3.3. *Reflection implies N_0.*

Proof. Recall our "strengthened" (but equivalent) version of Reflection:

$$\forall a \exists V_\beta (a \in V_\beta \wedge \forall y_1, y_2, \dots y_n \in V_\beta (\varphi(y_1, y_2, \dots, y_n) \leftrightarrow \varphi^{V_\beta}(y_1, y_2, \dots, y_n)))$$

Take $a = \varnothing$ and $\psi \equiv (\forall x \exists y\ (y = x \cup \{x\}) \wedge \varphi(x_1, \dots, x_n))$, and apply Reflection to ψ to obtain some V_β. Since φ holds in the universe φ^V holds in V_β. In particular for all ordinals $\alpha < \beta = V_\beta \cap \mathbf{On}$, we get $\alpha \in V_\beta$ so $\alpha + 1 \in V_\beta$. So β must be a limit cardinal. Now by Theorem 3.2, β a limit implies $\text{Scm}^{\text{S}}(V_\beta)$. □

For the reverse, we would like to show that $\text{Scm}^{\text{S}}(M)$ implies M is a stage V_λ for some limit, but without Replacement this is difficult. Instead we proceed via Infinity and Replacement (following Lévy's proofs), which we have already shown imply Reflection.

Theorem 3.4 (Infinity)**.** *N_0 implies $\exists y (\varnothing \in y \wedge \forall x \in y\ x \cup \{x\} \in y)$.*

Proof. Apply N_0 for $\varphi \equiv \exists x\ (x = \varnothing)$ to get a set M such that $\text{Scm}^S(M)$. Then M satisfies Union, Pairs, so $x^+ = x \cup \{x\}$ exists in M for all x (successors are absolute by Lemma 3.1). φ^M gives $\exists x \in M\ (x = \varnothing)$, so M is an inductive set. □

Theorem 3.5 (Replacement). *N_0 implies Replacement.*

Proof. It suffices to slightly modify the proof of Replacement from Scott's Reflection axiom in Theorem 2.15. The only properties of V we used were that V was transitive and V satisfied Comprehension. This is true of the set M obtained from N_0. What is missing in N_0 is the clause $a \in M$ for any fixed a. So for a class function $\varphi(u, v; x_1, \ldots, x_n)$ we let

$$\psi(a, x_1, \ldots, x_n) \equiv \forall x \in a \ \exists y \ \varphi(x, y; x_1, \ldots, x_n)$$

as before and apply N_0 to $(\varphi \wedge \psi \wedge \exists w \ (w = a))$. This relativises to $\exists w \in V \ w = a$ so $a \in V$. Then proceed exactly as in Theorem 2.15. □

So N_0 is an equivalent Reflection axiom to our original one, only it contains the statement that there exists a standard complete model for S. Further it is equivalent to Infinity and Replacement. Hence the jump from S to ZF, which we are usually happy to make, is equivalent to assuming that there is a model of ZF which reflects upward and downward with respect to the universe.

3.2 Large cardinals

The first historic breakthrough in set theory was Cantor's definition of ω. Not only do the natural numbers go on to infinity, but there exists a cardinal beyond them we cannot reach by our usual operations of plus, times and exponentiation (equivalently, Pairs, Unions and Power Set) applied to the natural numbers. If the stages should go on and on, one might argue, should there not be cardinals which are bigger than all the ordinals we can obtain by the methods ZFC provides? Supposing they exist, such cardinals are called *inaccessible*.

Inaccessible cardinals and other large cardinals play an important role in the modern study of set theory. In fact, inaccessible cardinals are the smallest in a hierarchy of ever-increasing cardinals – with each successive cardinal being so much larger that there are many cardinals of previous sort below it. (The sense of "many" depends on the precise type of large cardinal, as does the meaning of "large" (Kanamori, 2009, p. XXI)). All large cardinals offer increases in consistency strength over ZFC – in particular, their existence cannot be proved in ZFC due to Gödel's Incompleteness Theorems.

Besides establishing inner models of (parts of) set theory, large cardinals also establish properties of the real line, notably of perfect, Borel and analytic sets (known as *descriptive set theory*). They have applications in the theories of determinacy and infinitary combinatorics. Further they allow for solutions of new Diophantine equations (see e.g. Gödel, 1983, p. 477).

The existence of large cardinals gains support from *Maximise* as well as from the usefulness of their consequences. Kanamori and Magidor (1977, p. 103) argue that assuming large cardinals is similar to assuming the Axiom of Infinity (i.e. the existence of ω). We will show in this dissertation that α-inaccessible and α-Mahlo cardinals are also supported by Reflection by linking them to standard complete models of ZF.

The modern definition of inaccessible does not immediately relate to the idea of being "unreachable".

Definition 3.2. An infinite cardinal κ is *regular* if there exists no ordinal $\theta < \kappa$ and function $f : \theta \to \kappa$ such that $\sup \operatorname{ran} f = \kappa$. (Kunen (1980, p. 33) shows we may assume f is increasing.)
An infinite cardinal that is not regular is called *singular*.

For example, ω is regular, each successor cardinal $\aleph_{\alpha+1}$ is regular, but $\aleph_\omega$ is not as it is the limit of $\aleph_n$ for $n \in \omega$. (For detailed proofs, see Hrbacek and Jech (1999, pp. 161-162)).

Are there regular limit cardinals beyond ω? As we see with $\aleph_\omega$, regular limit cardinals are not so easily found. In fact, it is not possible to prove that uncountable regular limit cardinals exist. (This will follow from Gödel's Incompleteness Theorems applied to Section 3.3 below.)

Definition 3.3. A *weakly inaccessible cardinal* is a regular uncountable limit cardinal.

Definition 3.4. A *strongly inaccessible cardinal* is a regular uncountable strong limit cardinal κ, i.e. it is regular, uncountable and $2^\lambda < \kappa$ for all $\lambda < \kappa$.

We focus on strongly inaccessible cardinals, and following the usual trend refer to them simply as *inaccessible*. We proceed to define a hierarchy of them (modifying Kanamori's version for weak inaccessibles).

Definition 3.5. A 0-*strongly inaccessible cardinal* is a regular cardinal.
An $(\alpha+1)$-*strongly inaccessible cardinal* is a regular strong limit of α-strongly inaccessible cardinals, i.e. it is regular, a strong limit cardinal and a limit of α-strongly inaccessible cardinals.
A λ-*strongly inaccessible cardinal* (for λ a limit) is an α-strongly inaccessible cardinal for every $\alpha < \lambda$.

Note 1-inaccessibles are our original inaccessible cardinal: any strongly inaccessible cardinal is a limit cardinal and hence the limit of all successor cardinals before it, which are regular. In fact, we can condense the last two conditions to say κ is α-strongly inaccessible if it is regular and the limit of β-strongly inaccessible cardinals for each $\beta < \alpha$:

Lemma 3.6. *Let κ be an α-strongly inaccessible cardinal, then κ is β-strongly inaccessible for all $\beta \leqslant \alpha$. Hence if κ' is the limit of α-strongly inaccessible cardinals, it is the limit of β-strongly inaccessible cardinals for all $\beta \leqslant \alpha$.*

Proof. By induction. $\alpha = 0$ is vacuously true. If α is a limit, it is precisely the definition. Suppose $\alpha = \alpha' + 1$. Then κ is the limit of α'-inaccessible cardinals, say $\{\mu_\gamma\}_{\gamma<\kappa}$ (as κ is regular, the length of the sequence must be κ). Each μ_γ is β-inaccessible for all $\beta \leqslant \alpha'$ by inductive hypothesis, so κ is $\beta + 1$-inaccessible for all $\beta + 1 \leqslant \alpha' + 1$. Further κ is 0-strongly inaccessible as it is regular. □

We next discuss the model theory which gives inaccessible cardinals their name.

3.3 A similar step?

Consider now the following axiom schema:

Lévy's *N*. $\exists M(\mathrm{Scm}^{\mathrm{ZF}}(M) \wedge \forall x_1, \ldots, x_n \in u(\varphi(x_1, \ldots, x_n) \leftrightarrow \varphi^M(x_1, \ldots, x_n)))$.

Supposing ZF, now, we claim that there is a model of ZF which reflects upward and downward. Lévy argues that taking the step from ZF to ZF + N (ZN) is very similar to taking the step from S to ZF. That is,

> "we can view [the axiom schema N] as [a] natural continuation of the axioms of infinity and replacement... it seems likely that if in the sequence S, ZF, [ZN] , ... no inconsistency is introduced in the first step, from S to ZF, also no inconsistency is introduced in the further steps." (Lévy, 1960, p. 234)

Lévy then states that N is equivalent to the following axiom, referring to a result by Shepherdson (1952) that $\mathrm{Scm}^{\mathrm{ZF}}(M)$ holds iff $M = V_\kappa$ for some inaccessible κ.

Lévy's *N‴*. $\exists\alpha(\mathrm{In}(\alpha) \wedge \forall x_1, \ldots, x_n \in V_\alpha(\varphi(x_1, \ldots, x_n) \leftrightarrow \varphi^{V_\alpha}(x_1, \ldots, x_n)))$, where $\mathrm{In}(\alpha)$ abbreviates 'α is inaccessible'.

Note that the α depends on the formula φ. There are countably many formulas, so N assumes the existence of at most countably many inaccessible cardinals α. Compared to the number of ordinals, 'countably many' is vanishingly small.

However, the equivalence of these two axioms – on which Lévy bases the remainder of his argument – in fact requires rather strong assumptions. The backwards direction may be proved within ZFC, following Kanamori (2009). We present full versions of Kanamori's (rather brief) proofs.

Lemma 3.7 (Kanamori, 2009, Proposition 1.2a)**.** *Let κ be an inaccessible cardinal. If $x \subseteq V_\kappa$, then $x \in V_\kappa$ if and only if $|x| < \kappa$.*

Proof. We prove by induction on α that $|V_\alpha| < \kappa$ for $\alpha < \kappa$. Then since $x \in V_\kappa \to x \subseteq V_\alpha$ for some $\alpha < \kappa$ we have $|x| \leqslant |V_\alpha| < \kappa$.
Base case $\alpha = 0$: $|V_\alpha| = |\{\varnothing\}| = 1 < \kappa$.
$\alpha + 1$: $|V_{\alpha+1}| = |\mathcal{P}(V_\alpha)| = 2^{|V_\alpha|} < \kappa$ as $|V_\alpha| < \kappa$ by inductive hypothesis, and κ is a strong limit.
λ a limit: $|V_\lambda| = \left|\bigcup_{\alpha<\lambda} V_\alpha\right|$. For each $\alpha < \lambda$ $|V_\alpha| < \kappa$ by inductive hypothesis, so there is an injection $f_\alpha : V_\alpha \to \kappa$. Hence there is also an injection from $\bigcup_{\alpha<\lambda} V_\alpha$ to $\lambda \times \kappa$ by $x \mapsto (\alpha, f_\alpha(x))$ for $\alpha = \mathrm{rank}(x)$. But $|\lambda \times \kappa| = \lambda \cdot \kappa = \kappa$ as $\lambda < \kappa$, so $\left|\bigcup_{\alpha<\lambda} V_\alpha\right| < \kappa$.

Now suppose $x \subseteq V_\kappa$ with $|x| < \kappa$ and consider $f : x \to \mathbf{On}$ given by $y \mapsto \mathrm{rank}(y)$. By Replacement, f is a surjection onto the set $\mathrm{ran}\, f \subseteq \mathbf{On}$. By Choice (Cardinal Comparability), we have $|\mathrm{ran}\, f| \leqslant |x| < \kappa$. □

By Theorem 3.2, any V_λ for λ a limit is a model of S, so in particular for κ inaccessible. It remains (continuing with Scott's axiomatisation) to show Reflection.

Theorem 3.8. *Let κ be a (strongly) inaccessible cardinal. Then V_κ is a standard complete model of ZF.*

Proof. We want to show

$$\exists V_\alpha \in V_\kappa \forall x \in V_\alpha (\varphi(x) \to \varphi^{V_\alpha}(x))$$

We may follow the same proof as when we proved Lévy's Reflection Principle (Theorem 2.24), only we replace the use of Replacement by Lemma 3.7.

For each i such that φ_i is of the form $\exists t \ \varphi_j(v_1, \ldots, v_{n_i}, t)$ define $F_i : V_\kappa^{n_i} \to \kappa$ as in Theorem 2.24, replacing W with V_κ.

$$F_i(v_1, \ldots, v_{n_i}) = \begin{cases} 0 & \neg\exists t \in V_\kappa \ \varphi_j(w_1, \ldots, w_{n_i}, t)^{V_\kappa} \\ \eta & \eta < \kappa \text{ is least such that } \exists t \in V_\eta \ \varphi_j(w_1, \ldots, w_{n_i}, t)^{V_\kappa} \end{cases}$$

This is well-defined as κ is a limit cardinal.
As we are working in V_κ, **On** becomes $\mathbf{On} \cap V_\kappa = \kappa$. So G_i becomes $G_i : \kappa \to \kappa$, given by

$$G_i(\gamma) = \max\{\gamma + 1, \sup\{F_i(v_1, \ldots, v_{n_i}) : v_k \in V_\gamma\}\}$$

We need to check $\sup\{F_i(v_1, \ldots, v_{n_i}) : v_k \in V_\gamma\} \in V_\kappa$ so that G_i is well-defined. We have $F_i(v_1, \ldots, v_n) \in \kappa$ so certainly $F_i|_{V_\gamma^{n_i}} \subseteq \kappa$. Further

$$\left|F_i|_{V_\gamma^{n_i}}\right| \leqslant \left|V_\gamma^{n_i}\right| = n_i \cdot |V_\gamma| < n_i \cdot \kappa = \kappa$$

by Lemma 3.7 as $\gamma < \kappa$ so $V_\gamma \in V_\kappa$. By Lemma 3.7 $F_i|_{V_\gamma^{n_i}} \in V_\kappa$. Now, if $\left|F_i|_{V_\gamma^{n_i}}\right|$ is a successor cardinal, the set of ordinals $F_i|_{V_\gamma^{n_i}}$ has maximum $< \kappa$ since κ is a limit. If it is a limit ordinal, then it gives a sequence of length $\left|F_i|_{V_\gamma^{n_i}}\right|$ of ordinals in κ, so $\sup F_i|_{V_\gamma^{n_i}}$ must be less than κ by regularity of κ, so in V_κ.

Now define the sequence $\beta_{n+1} = \max_i G_i(\beta_n)$ and $\beta = \sup \beta_n$ as in the original proof. As all the β_n are values of G_i, so (β_n) is a sequence of length $\omega < \kappa$ of ordinals $< \kappa$, so by regularity as in the previous argument $\beta = \sup \beta_n < \kappa$. So V_β satisfies the condition of the Tarski-Vaught criterion as required. □

We can even show that Choice holds in V_κ (reverting to ZFC).

Theorem 3.9. *Let κ be an inaccessible cardinal. Then V_κ satisfies AC.*

Proof. f is a choice function for x if

$$f \in \mathcal{P}(\mathcal{P}(x) \setminus \{\varnothing\} \times x) \wedge \text{fun}(f) \wedge \forall a \in \mathcal{P}(x) \setminus \{\varnothing\} \ (f(a) \in a)$$

where $a \times b$ abbreviates $\{\{\{u\}, \{u, v\}\} : u \in a, v \in b\}$ and fun(f) abbreviates 'f is a function', i.e.

$$\forall x \in \mathcal{P}(x) \ \forall u \in x \ \forall v \in x \ (f(t) = u \wedge f(t) = v \to u = v))$$

The proofs above imply that V_κ satisfies Power Set, Union and Pairs, further all relevant notions are absolute. Hence being a choice function for x is absolute (the remaining parts are all Δ_0), so since Choice holds in the universe it also holds for all $x \in V_\kappa$. □

The reverse direction, however, cannot be proved in first-order logic: suppose we have a standard complete model M of ZF which satisfies $(\varphi \leftrightarrow \varphi^M)$ for some φ. Then by a modified version of the Downward Löwenheim-Skolem Theorem[2] there exists a transitive model $(M', \in)$ of ZF plus reflection on φ with the standard $\in$ relation which is countable. So it cannot possibly be a V_κ for κ inaccessible. In fact, since cardinals are not absolute, then without further conditions M (even with reflection) can say nothing about the existence of (inaccessible) cardinals in the universe at all.

This reflects the general issue that we cannot prove the categoricity of infinite theories in first-order logic. Instead, as for arithmetic and real analysis (see Shapiro, 1991), we resort to second-order logic.

The proof will proceed in two stages: first we show that M is of the form V_κ, then that κ is inaccessible.

Lemma 3.10. *If M is a standard complete model of ZF and F is an absolute class function defined on a set y, then $x = \bigcup_{z \in y} F(z)$ is absolute for M.*

Proof. By Replacement in M, the collection $A = \{F(z) : z \in y\}$ is a set in M. The formula '$x = \bigcup A$' is Δ_0 so absolute. □

Lemma 3.11. *If M is a standard complete model of ZF and the notion '$y = \mathcal{P}(x)$' is absolute for M then '$x = V_\alpha$' (for some x and ordinal α) is absolute for M.*

Proof. By transfinite induction on α.

$\alpha = 0$: '$x = V_0 = \varnothing$' is Δ_0 so absolute.
$\alpha = \beta + 1$: $x = V_{\beta+1}$ becomes $x = \mathcal{P}(V_\beta) \cup V_\beta$. By assumption and the Lemmas in Section 3.1 $x = \mathcal{P}(V_\beta) \cup V_\beta$ is absolute. Moreover the notion of '$\beta + 1$' is absolute as ordinals and successors are absolute.
$\alpha = \lambda$ for λ a limit: $x = V_\lambda$ becomes $x = \bigcup_{\beta<\lambda} V_\beta$ which is absolute by Lemma 3.10. □

Theorem 3.12. *Let M be a standard complete model of ZFC such that '$y = \mathcal{P}(x)$' is absolute for M. Then $M = V_\kappa$ for some ordinal κ.*

Proof. Recall M is a set, so $\kappa := \sup(\mathbf{On} \cap M)$ exists. κ must be a limit ordinal since by Pairs and Unions in M, $\alpha + 1 \in \mathbf{On} \cap M$ for all $\alpha \in \mathbf{On} \cap M$. We want to showthat $M = V_\kappa$.

By Lemma 3.11, the notion of stages is absolute for M and M satisfies the axioms required to build them. Hence for all $\beta \in \mathbf{On} \cap M$, i.e. for all $\beta < \kappa$, we must have $V_\beta \in M$. So $\bigcup_{\beta<\kappa} V_\beta \subseteq M$. Suppose $x \in M \setminus \bigcup_{\beta<\kappa} V_\beta$. We know x^+ is absolute by Lemma 3.1 and since Union and Pairs hold in M, $x^+ \in M$. Define the class function $F : y \mapsto \text{rank}(y)$ for $y \in x$. As $y \in x$, $\text{rank}(y) < \text{rank}(x) < \kappa$ so F is well-defined in M. Then by Replacement (in M) applied to F and x^+, $z = \{f(y) : y \in x^+\}$ is a set of ordinals in M. Hence $\text{rank}(x) \in z \in M$ and $z \in \mathbf{On} \cap M$ gives $z \subset \kappa$ so $\text{rank}(x) \in \kappa$ so $x \in V_\beta$ for some $\beta < \kappa$, contradiction. So $\bigcup_{\beta<\kappa} V_\beta = M$. □

When is '$y = \mathcal{P}(x)$' absolute? It need not be – for example it is absolute for V_λ for λ a limit by construction of stages, but not for Gödel's constructible stages L_λ (unless we

[2] In order for M' to be a standard complete model, it must be transitive and preserve $\in$. This is not guaranteed by the standard version of the theorem.

assume $V = L$). Certainly full second-order logic suffices, as then $\forall z \subseteq x(z \in y)$ is a bounded quantification over subsets (the other direction is Δ_0 so absolute even in first-order logic). By definition of second-order quantification, M and the universe must agree on $\forall z \subseteq x$ (assuming M is transitive).[3] The author has not been able to find discussions on precisely how much second-order logic is needed – the dominant position seems to be simply accepting that power sets are *not* absolute. The link between power sets and second-order logic in general is treated in Jané (2005).

For the second step, Kanamori (2009, p. 19) reduces the need for second-order logic to the assumption of second-order Replacement and that M satisfies Choice (which holds if M is an inaccessible stage V_κ by Theorem 3.9).

Definition 3.6 (Second-order Replacement)**.**

$$\forall F\ \forall x\ (\forall r((F(r,s) \wedge F(r,t)) \to s = t) \to \exists y \forall w(w \in y \leftrightarrow \exists v(v \in x \wedge F(v,w))))$$

If M is of the form V_α for some α, $\forall F$ relativises to $\forall F \subset V_\alpha \times V_\alpha$. If $x, y \in V_\alpha$ then $\{x\}, \{x,y\} \in V_{\alpha+1}$ so $\langle x,y\rangle \in V_{\alpha+2}$. Hence $F \subseteq V_{\alpha+2}$, so $F \in V_{\alpha+3}$. So we may replace the second-order quantification in V_α by first-order quantification in $V_{\alpha+3}$. It seems unlikely that we would accept the existence of some V_α but not of $V_{\alpha+3}$.

Theorem 3.13. *Let V_κ be a standard complete model of ZFC + second-order Replacement. Then κ is an inaccessible cardinal.*

Proof. κ is a regular cardinal: Suppose not, then we have a function $f : \beta \to \kappa$ such that $\{f(\gamma) : \gamma \in \beta\} = \kappa \notin M$, so M does not satisfy second-order Replacement, contradiction. (Crucially, Replacement must be second-order in order to make sure it covers this f. Otherwise f might not be definable in M by any formula φ – it might depend on sets M does not "know" about – but it is always a subset of M.)
κ is a strong limit: Suppose not, then there is a cardinal $\lambda < \kappa$ st. $2^\lambda \geqslant \kappa$ (using Cardinal Comparability, i.e. Choice, in M). That is, we have a surjection $g : \mathcal{P}(\lambda) \to \kappa$. $\mathcal{P}(\lambda) \in V_\kappa$ since Power Set holds in V_κ. So by second-order Replacement, $\operatorname{ran} g = \kappa \in V_\kappa$, contradiction. □

However, Lévy's similarity argument relies on M being a model of ZF alone, not of ZFC.

Lévy cites an older proof by Shepherdson (1952) that $\mathrm{Scm}^{\mathrm{ZF}}(M) \Leftrightarrow \exists \kappa\ M = V_\kappa$ for κ inaccessible. Shepherdson's paper is couched in full second-order logic and moreover assumes Global Choice in the universe in order to prove this theorem. The precise assumptions required are Global Choice and that the following notions are absolute:

1. 'f is a function on an ordinal α'
2. The range of a function (on an ordinal)
3. (Infinite) unions
4. Cardinality of sets
5. Power sets of cardinals

[3] Lévy refers to Shepherdson (1952), who refers to Shepherdson (1951) for this result. However there either is a typographical error in Shepherdson's paper or this result is missing, as there is no such proposition '2.236'.

Absoluteness of cardinals in particular cannot follow from first-order logic due to the Löwenheim-Skolem Theorems. (It would be an interesting question for further research to determine precisely how much second-order logic is needed for Shepherdson's proof, and whether Global Choice is strictly necessary.)

In light of these assumptions, recall Lévy's original claim: that N and N''' are natural continuations of N_0, i.e. the assumption of Infinity and Replacement. While this may be true of N, it cannot said lightly of N''', since N''' requires not only a standard complete model of ZF (as N_0 did of S) but also significant portions of second-order logic and some form of Choice. We even need Choice to show N from N''' (in Lemma 3.7).

Lévy avoids this in part by redefining the notion of 'inaccessible' at the start of his paper to a weaker form which can be proved from N without Choice. In his introduction, he also sets out his system as "non-simple applied first-order functional calculus" (p. 223) – that is, he allows functions as variables, which range over "all subsets of the universe set of the model" (p. 224). His version of Replacement (misleadingly) appears first-order, as Lévy only builds the quantification over all functions into his definition of standard complete model ("in the sense of Henkin") some paragraphs later.[4]

While technically correct, these (re-)definitions obfuscate what Lévy actually proves. He refers often to inaccessible cardinals and to ZF, yet his definition of 'ZF' is much stronger than what modern mathematicians assume, while his definition of 'inaccessible' is weaker. His assertion that the step to N''' is a natural continuation of the step from S to ZF relies on ZF being second-order already.

It is questionable whether second-order Replacement is as intuitive as the first-order Replacement and Infinity commonly assumed in ZF. Quine (1986, pp. 64-68) famously highlights the difficulties of second-order logic regarding lack of completeness, ontological commitment, and overlap with set theory.[5] Further, soundness of second-order logic only follows given suitable semantics. While counterarguments to the first two points can be found in Boolos (1975) and Shapiro (1991) provides a highly detailed analysis and rebuttal of all four, the issue remains a delicate one and needs to be treated with some caution.[6]

None of this falsifies Lévy's mathematics, but it casts serious doubt on Lévy's assertion that assuming N''' is so similar to N_0 and "as unlikely to introduce inconsistency" (p. 234). Reviewers of Lévy's paper seem to have focused (as best as the author can determine) on the mathematical content of his paper rather than critiquing his more philosophical assertions.

In the remainder of the paper, we give some results that follow from the assumption of N''' and strengthenings thereof. These are intended to motivate further research to fill the gaps in Lévy's justification of N'''.

[4]For a discussion of Henkin semantics, see Shapiro (1991).

[5]For example, we can write down a second-order sentence that holds if and only if the Continuum Hypothesis is true, and moreover this sentence is absolute between transitive models.

[6]Shapiro also argues that our modern fixation on first-order logic is merely a remnant of Hilbert's foundationalist programmes which have been shown to fail. Whether this is indeed justified and whether it applies to the topics of this dissertation could be the subject of further research.

3.4 A stronger result

A highlight of Lévy's paper is his proof that N''' suffices for the existence of unbounded α-inaccessible cardinals. That is, to show their existence we need "just" countably many inaccessible cardinals satisfying Reflection.

Definition 3.7. A *normal function* is a class function $F : \mathbf{On} \to \mathbf{On}$ (or a function $f : A \to B$ for $A, B \subset \mathbf{On}$) which is strictly increasing, i.e. $\alpha < \beta \to F(\alpha) < F(\beta)$, and continuous, i.e. for limit λ, $F(\lambda) = \sup_{\beta<\lambda} F(\beta)$.

Lemma 3.14. *For every normal function F defined for all ordinals, $F(\alpha) \geqslant \alpha$.*

Proof. By transfinite induction. $F(0) \geqslant 0$ is clear. $F(\alpha + 1) > F(\alpha) \geqslant \alpha$ by inductive hypothesis so $F(\alpha + 1) \geqslant \alpha + 1$. For λ a limit, $F(\lambda) = \sup_{\beta<\lambda} F(\beta) \geqslant F(\beta) \geqslant \beta$ for all $\beta < \lambda$, so $F(\lambda) \geqslant \sup_{\beta<\lambda} \beta = \lambda$. □

Lemma 3.15. *Every normal function F has arbitrarily large fixed points.*

Proof. Fix $\alpha \in \mathbf{On}$ and construct the following sequence: For $n \in \omega$, let $\beta_0 = \alpha$, $\beta_{n+1} = F(\beta_n)$. Let $\beta = \sup_{n\in\omega} \beta_n$. We have $\beta \geqslant \alpha$ and $F(\beta) = F(\sup_{n\in\omega} \beta_n) = \sup_{n\in\omega} F(\beta_n) = \sup_{n\in\omega} \beta_{n+1} = \beta$ (as β is a limit ordinal). Hence β is a fixed point larger than our arbitrary α. □

We use normal functions to enumerate the α-inaccessible cardinals. Let $P_\alpha(0)$ be the first α-inaccessible cardinal, supposing it exists, let $P_\alpha(\xi + 1)$ be the first α-inaccessible cardinal greater than $P_\alpha(\xi)$, supposing it exists, and to get continuity we define $P_\alpha(\lambda) = \sup_{\xi<\lambda} P_\alpha(\xi)$ for λ a limit (note $P_\alpha(\lambda)$ need not be α-inaccessible itself). We do not yet suppose that P_α is defined on all ordinals or for all α, we will show this later.

Consider the following statement:

Lévy's *M*. Every normal function defined for all ordinals has at least one inaccessible cardinal in its range.

Theorem 3.16 (Lévy, 1960, Theorem 1)**.** *Let F be a normal function defined for all ordinals. The following are equivalent:*

M: F has at least one inaccessible cardinal in its range.

M': F has at least one fixed point which is inaccessible.

M'': F has arbitrarily great fixed points which are inaccessible.

Proof. Clearly $M'' \Rightarrow M' \Rightarrow M$. We will prove $M \Rightarrow M''$.
Let F be a normal function defined for all ordinals. Let G be the normal function enumerating its fixed points, i.e. $G(0)$ is the least β st. $F(\beta) = \beta$, $G(\alpha + 1)$ is the least $\beta > G(\alpha)$ such that $F(\beta) = \beta$, and $G(\lambda) = \sup_{\beta<\lambda} F(\beta)$ (for λ a limit). By Lemma 3.15, G is also defined for all ordinals Fix any λ, define $H_\gamma(\xi) = G(\gamma + \xi)$. H_γ is also a normal function defined for all ordinals (as G is), so by M there is an ordinal such that $\beta = H_\gamma(\xi)$ is inaccessible. $\beta = G(\gamma + \xi)$ so by definition of G $F(\beta) = \beta$. By Lemma 3.14, $\beta \geqslant \gamma + \xi \geqslant \gamma$. □

In particular, M implies that there are arbitrarily large inaccessible cardinals, i.e. the class of inaccessible cardinals is unbounded. Lévy shows[7] that we can prove M from N'''.

Theorem 3.17 (Lévy, 1960, Theorem 3)**.** *N''' implies M.*

Proof. Let $\varphi(x, y, v_1, \ldots, v_n)$ be a formula. Let $\chi(v_1, \ldots, v_n)$ be the formula asserting that if $\varphi(\xi, \eta, v_1, \ldots, v_n)$ gives a normal function $\eta = F(\xi)$ defined for all ordinals then F has at least one inaccessible cardinal in its range. (That is, χ is equivalent to the statement that M holds for this φ.) Let $\rho(v_1, \ldots, v_n)$ be the formula asserting that F is a normal function. Consider now $\Phi = \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4$, where

$$\begin{aligned} \varphi_1 &\equiv \varphi \\ \varphi_2 &\equiv \forall\xi\exists\eta\varphi(\xi, \eta) \wedge \rho \\ \varphi_3 &\equiv \chi \\ \varphi_4 &\equiv \forall v_1, \ldots, v_n\chi \end{aligned}$$

That is, φ_2 states that φ is defined for all ordinals. We now apply N''' to Φ to get (as relativisation passes through $\wedge$) that there exists some inaccessible α such that:

$$v_1, \ldots, v_n \in V_\alpha \to (\varphi \leftrightarrow \varphi^{V_\alpha}) \tag{3.1}$$

$$v_1, \ldots, v_n \in V_\alpha \to ((\forall\xi\exists\eta\ \varphi(\xi, \eta) \wedge \rho) \leftrightarrow (\forall\xi < \alpha\exists\eta < \alpha\ \varphi(\xi, \eta)^{V_\alpha} \wedge \rho^{V_\alpha})) \tag{3.2}$$

$$v_1, \ldots, v_n \in V_\alpha \to (\chi \leftrightarrow \chi^{V_\alpha}) \tag{3.3}$$

$$(\forall v_1, \ldots, v_n\ \chi) \leftrightarrow (\forall v_1, \ldots, v_n \in V_\alpha\ \chi^{V_\alpha}) \tag{3.4}$$

To prove M, we assume that for some $v_1, \ldots, v_n \in V_\alpha$, $\varphi(\xi, \eta)$ gives a normal function defined for all ordinals (In the other cases M is vacuously true.) Since we have assumed the left hand side of (3.2), we get

$$\forall\xi < \alpha\ \exists\eta < \alpha\ \varphi(\xi, \eta)^{V_\alpha}$$

By (3.1), φ holds iff its relativisation holds, so we have

$$\forall\xi < \alpha\ \exists\eta < \alpha\ \varphi(\xi, \eta)$$

We claim $F(\alpha) = \alpha$. To see this, consider $\beta < \alpha$. By (3.2) $F(\beta) \in V_\alpha$ i.e. $F(\beta) < \alpha$. α is a limit so $F(\alpha) = \sup_{\beta<\alpha} F(\beta) \leqslant \alpha$. But $F(\alpha) \geqslant \alpha$ so $F(\alpha) = \alpha$.
So F has at least one inaccessible cardinal in its range. This proves $v_1, \ldots, v_n \in V_\alpha \to \chi$ so by (3.3) we have $v_1, \ldots, v_n \in V_\alpha \to \chi^{V_\alpha}$ which is equivalent to $(\forall v_1, \ldots, v_n\chi)^{V_\alpha}$. By (3.4) this is equivalent to $\forall v_1, \ldots, v_n\chi$, thus proving M. □

In fact, Lévy shows that we can prove the existence of unbounded α-inaccessible cardinals – a much stronger result. For part of the argument, Lévy refers his reader to Mahlo's original paper *Über lineare transfinite Mengen* (1911). We offer a combined (and translated) version of the two proofs here, and add the case for $\xi = 0$ which Lévy omits.

Rephrasing the definition of $(\alpha + 1)$-inaccessible cardinals gives:

Lemma 3.18. *The limit of a sequence of α-inaccessible cardinals of length γ, where γ is regular, is either an $(\alpha + 1)$-inaccessible cardinal or not even a regular cardinal.*

[7] In fact, he shows M and N''' are equivalent

Recall we defined $P_\alpha(\xi)$ to be the normal function enumerating (in increasing order) the α-inaccessible cardinals, insofar as they are defined.

Theorem 3.19 (Lévy, 1960, Theorem 4)**.** *Assuming M, then $P_\alpha(\xi)$ is defined for all α, ξ. In other words, the α-inaccessible cardinals are unbounded for each $\alpha \in \mathbf{On}$.*

Proof. Let α be the least ordinal such that P_α is not defined for all ordinals and let ξ be the least ordinal for which $P_\alpha(\xi)$ is not defined. ξ cannot be a limit as otherwise we could define $P_\alpha(\xi) := \sup_{\eta<\xi} P_\alpha(\eta)$. Moreover $\alpha > 0$: for all β there is some successor cardinal greater than β and all successors are regular, so $P_0(\xi)$ is defined for all ordinals.

Case 1: $\alpha = 1$, i.e. P_α enumerates the inaccessible cardinals. By M there exists an inaccessible cardinal so $\xi > 0$. Since ξ is not a limit, write $\xi = \xi' + 1$. Let $\beta = P_1(\xi')$. Define $F(0)$ to be the least regular cardinal $> \beta$, which is $P_0(\gamma)$ for some γ, and let $F(\eta)$ enumerate the regular cardinals $> \beta$, i.e. $F(\eta) = P_0(\gamma + \eta)$. Since P_0 is defined for all ordinals by the above, so is F. By M it has an inaccessible cardinal in its range, but $F(\eta) > \beta$ for all η, contradiction as $\beta = P_1(x')$ was the greatest inaccessible cardinal.

Case 2: $\alpha > 1$ is a successor ordinal. If $\xi = 0$, let $\beta = 0$, else write $\xi = \xi' + 1$ and let $\beta = P_\alpha(\xi')$. Since $\alpha = \alpha' + 1$ is least, $P_{\alpha'}(\eta)$ is defined for all η. Define $F(0)$ to be the least α'-inaccessible cardinal $> \beta$, which is $P_{\alpha'}(\gamma)$ for some γ, and let $F(\eta)$ enumerate the α'-inaccessible cardinals $> \beta$, i.e. $F(\eta) = P_{\alpha'}(\gamma + \eta)$. Since $P_{\alpha'}$ is defined for all ordinals so is F. By M', F must have an inaccessible fixed point, i.e. some inaccessible $F(\mu) = \mu$. As μ is inaccessible, μ is also a limit. By Lemma 3.18, $F(\mu) = \sup_{\lambda<\mu} F(\lambda)$ is either α-inaccessible or not even a regular cardinal. However $F(\lambda) > \beta = P_\alpha(\xi')$ for all λ so since $P_\alpha(\xi')$ is the greatest α-inaccessible cardinal (or $\beta = 0$ and there are no α-inaccessibles), $F(\mu)$ cannot be α-inaccessible. Hence $F(\mu) = \mu$ is not regular, so not inaccessible, contradiction.

Case 3: $\alpha > 1$ is a limit ordinal. As in the previous case, if $\xi = 0$, let $\beta = 0$, else write $\xi = \xi' + 1$ and let $\beta = P_\alpha(\xi')$. Define $\gamma_{\varepsilon\mu}$ for $\varepsilon < \alpha$ and $\mu \in \mathbf{On}$ as follows: Let γ_{00} be the least regular cardinal $> \beta$, let $\gamma_{\varepsilon\mu}$ be the least ε-inaccessible cardinal such that $\gamma_{\varepsilon\mu} > \gamma_{\varepsilon'\mu'} > \beta$ for all $\varepsilon' < \varepsilon, \mu' < \mu$.

$$\begin{matrix} \gamma_{00} & \gamma_{10} & \cdots & \gamma_{\varepsilon 0} & \cdots \\ \gamma_{01} & \gamma_{11} & \cdots & \gamma_{\varepsilon 1} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ \gamma_{0\mu} & \gamma_{1\mu} & \cdots & \gamma_{\varepsilon\mu} & \cdots \\ \vdots & \vdots & & \vdots & \end{matrix}$$

Let G enumerate the $\gamma_{\varepsilon\mu}$ starting first with the $\gamma_{\varepsilon 0}$ (for $\varepsilon < \alpha$), then the $\gamma_{\varepsilon 1}$ and so forth through all the $\mu \in \mathbf{On}$. By M', G must have an inaccessible fixed point $G(\nu) = \nu$, where ν is a limit ordinal. Consider $G(\nu) = \sup_{\lambda<\nu} G(\lambda)$. Suppose $\nu \neq \kappa \cdot \alpha$ for some κ. Then $G(\nu)$ is the limit of the sequence

$$\gamma_{00} \cdots \gamma_{\varepsilon 0} \cdots \gamma_{01} \cdots \gamma_{\varepsilon 1} \cdots \gamma_{0\mu} \cdots \gamma_{\varepsilon'\mu}$$

for some μ, ε' such that $\mu \cdot \alpha + \varepsilon' = \nu$. Equivalently, $G(\nu)$ is the limit of the "last row" $\gamma_{0\mu} \ldots \gamma_{\varepsilon'\mu}$ of length $\varepsilon' < \alpha$. Moreover $G(\nu) > \gamma_{00} > \beta > \alpha$, so $G(\nu)$ is the limit of less than α cardinals $< \alpha$, so cannot be regular, let alone inaccessible, contradiction.

Suppose instead $\nu = \kappa \cdot \alpha$ for some κ. Each $\gamma_{\varepsilon\mu}$ is ε-inaccessible, so for each $\varepsilon < \alpha$, $G(\nu)$ is the limit of κ many ε-inaccessible cardinals (the "columns"). Hence since $G(\nu)$ is inaccessible then $G(\nu)$ is α-inaccessible by definition, but $G(\nu) > \beta$, contradiction, as either $\beta = P_\alpha(\xi')$ was the greatest α-inaccessible cardinal or if $\beta = 0$ there were no α-inaccessible cardinals. □

3.5 A hierarchy of reflection principles

In our definition of N we introduced cardinals which were inaccessible with respect to ZF. What about cardinals which are inaccessible with respect to ZN? Such cardinals are called *Mahlo cardinals*. As for inaccessible cardinals, we state the modern definition and show it gives inaccessibility from ZN.

Definition 3.8. Let κ be a regular uncountable cardinal. A set C is *unbounded* in κ if $\sup C = \kappa$. A set $C \subseteq \kappa$ is *closed* if for all $\gamma < \kappa$ such that $C \cap \gamma$ is unbounded in γ, $\gamma \in C$.

We may think of closed unbounded sets as being "large" or having (probabilistic) "measure 1".[8]

Definition 3.9. Let κ be a regular uncountable cardinal. A set S is *stationary* (in κ) if for all closed unbounded sets $C \subseteq \kappa$, $S \cap C$ is nonempty.

We may think of stationary sets as having "non-zero measure".

Definition 3.10. A *weakly Mahlo cardinal* κ is a regular uncountable cardinal such that the set of regular cardinals $< \kappa$ is stationary in κ.
A *strongly Mahlo cardinal* κ is a weakly Mahlo and strongly inaccessible cardinal.

As we focused on strongly inaccessible cardinals, we focus on strongly Mahlo cardinals. We follow Kanamori (2009, p. 17) in building up a hierarchy of them, modifying his 0-weakly Mahlo cardinals to make 0-strongly Mahlo cardinals inaccessible rather than simply regular.

Definition 3.11. A 0-*strongly Mahlo cardinal* is a strongly inaccessible cardinal.
An $(\alpha + 1)$-*strongly Mahlo cardinal* is a strongly inaccessible cardinal κ such that the set of α-strongly Mahlo cardinals $< \kappa$ is stationary in κ.
A λ-*strongly Mahlo cardinal* (for λ a limit) is an α-strongly Mahlo cardinal for all $\alpha < \lambda$.

As for inaccessible cardinals, we henceforth omit 'strongly'. Again we can prove that if κ is α-strongly Mahlo then it is β-strongly Mahlo for all $\beta < \alpha$. So we can condense the successor and the limit case to say that κ is α-strongly Mahlo iff the set of β-strongly Mahlo cardinals is stationary in κ for all $\beta < \alpha$.

Kanamori (2009, Proposition 1.1) shows that for $\alpha > 0$, every α-strongly Mahlo cardinal κ is κ-inaccessible. This confirms why we consider Mahlo cardinals to be "larger" than inaccessible cardinals, in addition to there being "many" inaccessible cardinals below each Mahlo cardinal (expressed by them being stationary).

[8]For example, the intersection of closed unbounded sets is closed unbounded, similar to measure 1 sets.

Lévy and Mahlo use a different definition of Mahlo cardinals, in line with axiom M.

Definition 3.12 (Lévy, 1960, p. 233)**.** A 1-*strongly Mahlo cardinal* is an inaccessible cardinal κ such that each normal function $f : \kappa \to \kappa$ has at least one inaccessible cardinal in its range. An $(\alpha+1)$-*strongly Mahlo cardinal* is an inaccessible cardinal κ such that each normal function $f : \kappa \to \kappa$ has at least one α-strongly Mahlo cardinal in its range. A λ-*strongly Mahlo cardinal* (for λ a limit) is an α-strongly Mahlo cardinal for all $\alpha < \lambda$.

We prove that these two definitions are equivalent for $\alpha > 0$ by the following well-known conversion.

Lemma 3.20 (Equivalence of closed unbounded sets and normal functions)**.** *Let κ be an uncountable regular cardinal.*

1. *If $f : \kappa \to \kappa$ is a normal function then* $\operatorname{ran} f$ *is closed unbounded in κ.*
2. *If $C \subseteq \kappa$ is closed unbounded in κ then the function $f : \alpha \mapsto (\alpha^{th}$ element of $C)$ is normal, moreover f is a function from κ to κ.*

Lemma 3.21. *κ is α-strongly Mahlo for $\alpha > 0$ (by our original definition) iff κ is an inaccessible cardinal such that each normal function $f : \kappa \to \kappa$ has at least one β-strongly Mahlo cardinal in its range for each $\beta < \alpha$.*

Proof. κ is $(\alpha+1)$-strongly Mahlo iff it is inaccessible and the set S of α-strongly Mahlo cardinals is stationary in κ. Let $f : \kappa \to \kappa$ be any normal function. Then $\operatorname{ran} f$ is closed unbounded in κ, so $\operatorname{ran} f \cap S \neq \varnothing$.
Conversely suppose every normal function on κ has at least one α-strongly Mahlo cardinal in its range. Consider a closed unbounded set $C \subseteq \kappa$. By the previous Lemma obtain a normal function $f : \kappa \to C$, by assumption this has an α-inaccessible cardinal in its range C. So $C \cap S \neq \varnothing$.
If α is a limit, pick $\beta < \alpha$. By definition κ is $(\beta+1)$-strongly Mahlo so each normal function has at least one β-strongly Mahlo cardinal in its range. □

Using these definitions, Lévy (1960, p. 233) defines the axiom schemata N_Λ and N'''_Λ for any definable[9] ordinal $\Lambda > 0$ as follows.

Lévy's N'''_Λ. $\forall\mu < \Lambda \exists\kappa(\kappa$ is μ-Mahlo $\wedge\, \forall x_1, \ldots, x_n \in V_\kappa(\varphi \leftrightarrow \varphi^{V_\kappa}))$.

For $\Lambda' = \Lambda + 1$ a successor ordinal, this is equivalent to

Lévy's $N'''_{\Lambda+1}$. $\exists\kappa(\kappa$ is Λ-Mahlo $\wedge\, \forall x_1, \ldots, x_n \in V_\kappa(\varphi \leftrightarrow \varphi^{V_\kappa}))$.

since Λ-strongly Mahlo implies μ-strongly Mahlo for all $\mu \leqslant \Lambda$. Hence for $\Lambda = 1$, this is equivalent to our original N as $\mu < 1$ means κ is 0-strongly Mahlo, i.e. inaccessible.

We can also consider

Lévy's M_Λ. Every normal function defined for all ordinals has for every $\mu < \Lambda$ at least one μ-Mahlo cardinal in its range.

Again M_1 is our original M. Further by extending Lemma 3.20 and the definitions of stationary sets to classes, M_Λ can be written as "The class of μ-Mahlo cardinals is

[9] If we take 'definable' as recursively definable, then Λ is bounded above by the Church-Kleene ordinal ω_1^{CK}, and hence countable. (This comment is due to Robert Leek.) In any case we assume that the existence of Λ is provable in ZF to avoid technicalities.

(definably) stationary in **On** for all $\mu < \Lambda$", i.e. every definable closed unbounded class of ordinals contains a μ-Mahlo cardinal (Kanamori, 2006).

Let $\mathrm{ZM}_\mu = \mathrm{ZF} + M_\mu$. Building on Theorem 3.12 we quickly get:

Theorem 3.22. *If κ is Λ-Mahlo, then V_κ satisfies ZM_Λ for all $\mu < \Lambda$, conversely if $Scm^{\mathrm{ZM}_\mu}(M)$ for all $\mu < \Lambda$ then $M = V_\kappa$ where κ is Λ-Mahlo.*

Proof. Suppose κ is Λ-Mahlo. In particular κ is inaccessible so by Theorem 3.8 V_κ satisfies ZF. To prove M_Λ (and hence M_μ for all $\mu < \Lambda$), we need to show that every normal function $f : \kappa \to \kappa$ has for every $\mu < \Lambda$ at least one μ-Mahlo cardinal in its range ('defined for all ordinals' relativises to $< \kappa$). But by Lemma 3.21 this is precisely the definition of Λ-Mahlo.
Suppose M is a standard complete model of ZM_Λ. By Theorem 3.12 $M = V_\kappa$ for some inaccessible κ. Since M satisfies M_Λ, then every function $F : \kappa \to \kappa$ has for every $\mu < \Lambda$ at least one μ-strongly Mahlo cardinal in its range. So by Lemma 3.21 κ is Λ-Mahlo. ☐

In particular, if κ is 1-Mahlo then V_κ is a model of ZM. So this theorem proves that Mahlo cardinals are inaccessible with respect to ZM and ZN.

This also allows us to define

Lévy's N_Λ. $\forall \mu < \Lambda \exists u(\mathrm{Scm}^{\mathrm{ZM}_\mu}(u) \wedge \forall x_1, \ldots, x_n \in V_\kappa(\varphi \leftrightarrow \varphi^u)$.

Hence we are "reflecting reflection", by repeatedly applying reflection to our system just obtained. Thus we get a whole sequence S, ZF, ZN, ZN_2, …, ZN_Λ, … of increasingly strong axioms, where each step is comparable to the last. (As discussed in Section 3.3, these steps are no longer similar when we replace N with N'''.)

By Theorem 3.22, N'''_Λ and N_Λ are equivalent – assuming the same (significant) amount of second-order logic and Choice as in Theorem 3.12. Further Lévy claims that "by complete analogy" to N''' and M, N'''_Λ and M_Λ are equivalent.[10] Indeed upon examining the proof of Theorem 3.17 we see that we may replace 'inaccessible' by 'μ-Mahlo' for each μ and the proof will run identically, since N'''_Λ gives the existence of a μ-Mahlo cardinal κ (and corresponding stage V_κ) which is precisely the μ-Mahlo cardinal in the range of the normal function we begin the proof with.

From M_Λ it follows easily that the μ-inaccessible cardinals are unbounded for each $\mu < \Lambda$ (the result mimics Theorem 3.19, but M_Λ is substantially stronger than the statement we are proving).

Lemma 3.23. *Given M_Λ, the normal function $R_\mu(\xi)$ enumerating the μ-Mahlo cardinals is defined for all $\mu < \Lambda$ and for all ordinals ξ.*

Proof. Let μ be the least ordinal $< \Lambda$ for which R_μ is not defined for all ordinals and let ξ be the least ordinal for which $R_\mu(\xi)$ is not defined. By M_Λ, there exists at least one μ-Mahlo cardinal so $\xi > 0$. As in Theorem 3.19 ξ cannot be a limit since otherwise we could define $R_\mu(\xi)$ (since R is a normal function). Write $\xi = \xi' + 1$.
Let $\beta = R_\mu(\xi')$. Let $F(\eta) = \beta + (\eta + 1)$. This is defined for all ordinals, so by M_Λ has a μ-Mahlo cardinal in its range. So there exists a μ-Mahlo cardinal greater than $\beta = R_\mu(\xi')$, contradiction. ☐

[10]Refer to Lévy (1960, Theorem 3) for the proof that M implies N'''.

A more interesting question would be whether we can deduce the existence of μ-Mahlo cardinals from M_Λ for some $\Lambda < \mu$, as we did in Theorem 3.19 for inaccessible cardinals. Lévy does not pursue this avenue any further, merely outlining the correspondence between N_Λ and M_Λ. This would also require further development of Mahlo's methods used in the proof of Theorem 3.19.

Even without such results, the existence of many μ-Mahlo cardinals follows immediately from each M_Λ (for $\mu < \Lambda$), which in turn follow from N'''_Λ. Hence if we can remedy the gaps in Lévy's argument to show that the step from N to N''' (and hence also from N_Λ to N'''_Λ) is justified, we may build our hierarchy of axioms N and M and deduce the existence of unbounded α-inaccessible and α-Mahlo cardinals from them.

Chapter 4

Conclusion

We started out by examining the roots of set theory and proposing that we ground set theory in the iterative conception. The iterative conception lends itself to an intuitive and neat axiomatisation containing a reflection principle. Reflection principles are well-supported – both as theorems proved in ZFC and as a guideline for viewing the universe of sets and introducing new axioms. We then critically discussed Lévy's claim that in a very similar step to adding Reflection to S (ZF without Infinity and Replacement), we could strengthen our Axiom of Reflection to give the existence of inaccessible cardinals. From this new principle we deduced the existence of unbounded classes of inaccessible and α-inaccessible cardinals. Strengthening Reflection further to a hierarchy of reflection principles, we gained the existence of Mahlo cardinals.

The aim of this dissertation has been to present the justification leading from the iterative conception via reflection principles to large cardinals. The crucial and most fragile step is the one from standard complete models to inaccessible cardinals, which requires a significant amount of second-order logic. We concluded that Lévy's arguments were misleading as they relied on the unconditional assumption of second-order Replacement as part of ZF. Nevertheless, Lévy's axioms are powerful and have very desirable consequences: in particular the existence of inaccessible cardinals gives us the consistency of ZFC and supports the use of Grothendieck universes in algebra (see e.g. McLarty, 2010). Hence further research on the justification of Lévy's argument and of reflection principles in general is certainly warranted.

4.1 Larger cardinals

A number of other types of large cardinal have been linked to (increasingly strong) reflection principles, and will gain support if we can further develop our justification of reflection.

Examples include *indescribable* cardinals, which are defined in terms of reflection principles and developed by Hanf and Scott (1961). These provide a systematic approach to comparing other large cardinals by consistency strength (Kanamori, 2009, p. XVIII).

Measurable and *extendible* cardinals have also been shown to follow from a strong form of Reflection by Reinhardt (1974). Measurable cardinals are an important current field of study, relating closely to descriptive set theory. For example, certain kinds of measurable cardinal imply that there is a countably additive extension of the Lebesgue measure to the whole real line (Kanamori, 2009, p.24). Further still, supercompact cardinals can be expressed in terms of a global reflection property (Kanamori, 2009, p.299). As noted above, the reflection principles used to justify measurable cardinals in Reinhardt's paper are very strong. Wang (1983, p. 555) argues that they are in fact too strong to be implied by even a maximal iterative conception of sets. (The same may turn out to be true of the reflection principles needed for all larger cardinals.)

Finally Woodin cardinals (slightly weaker than supercompact cardinals) tie in with the study of determinacy, another central branch of modern set theory, as shown by Woodin in 1985 (see Kanamori, 2009, p. 464). Kanamori suggests that we can frame Woodin cardinals as a "kind of reflection property" (2009, p. 364). If we can link these large cardinals firmly to reflection principles, this will shed further light on prospective axiom candidates like the Axiom of Projective Determinacy, which notably implies the Lebesgue measurability of all projective subsets of the reals.

Alternatively, we can extend reflection to second-order logic. Second-order reflection principles as postulated e.g. by Bernays have many strong implications, including Global Choice (Kanamori, 2009, p. 59).

All in all, Reflection principles remain a promising tool for both exploring and justifying the higher infinite.

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Appendix A

Reflection in n variables

Theorem 2.14 (Reflection in n variables)**.**

$$\forall a \exists V (a \in V \wedge \forall y_1, y_2, \dots y_n \in V(\varphi(y_1, y_2, \dots, y_n) \leftrightarrow \varphi^V(y_1, y_2, \dots, y_n)))$$

Proof. Let

$$\psi_1(x) \equiv \forall y_1, \dots, y_n, z(x = \langle y_1, \dots, y_n, z\rangle \to (\varphi(y_1, \dots, y_n) \leftrightarrow z = 0))$$

Let

$$\psi_2 \equiv \forall y_1, \dots, y_n, z \exists w \ (w = \langle y_1, \dots, y_n, z\rangle)$$

This holds by the Axiom of Pairs in the universe.
Let

$$\psi_3 \equiv \exists u \ (u = a)$$

for some fixed (constant) a given above. This is clearly true in the universe.
Their relativisations are:

$$\begin{aligned}
&\psi_1^V \equiv \forall y_1, \dots, y_n, z \in V(x = \langle y_1, \dots, y_n, z\rangle \to (\varphi^V(y_1, \dots, y_n) \leftrightarrow z = 0)) \\
&\psi_2^V \equiv \forall y_1, \dots, y_n, z \in V \exists w \in V \ (w = \langle y_1, \dots, y_n, z\rangle) \\
&\psi_3^V \equiv \exists u \in V \ (u = a)
\end{aligned}$$

Let $\psi \equiv \psi_1 \wedge \psi_2 \wedge \psi_3$. Note that by the above $\psi \leftrightarrow \psi_1$. By Reflection applied to ψ, we have

$$\begin{aligned}
& \exists V \forall x \in V((\psi_1(x) \wedge \psi_2 \wedge \psi_3) \to (\psi_1^V(x) \wedge \psi_2^V(x) \wedge \psi_3^V(x))) \\
\Leftrightarrow \quad & \exists V \forall x \in V(\psi_1(x) \to (\psi_1^V(x) \wedge \psi_2^V(x) \wedge \psi_3^V(x))) \qquad \text{(A.1)}
\end{aligned}$$

Choose the V given above, fix $x, y_1, \dots, y_n, z \in V$ and suppose $\psi_1(x)$ and $x = \langle y_1, \dots, y_n\rangle$. Then $\varphi(y_1, \dots, y_n) \leftrightarrow z = 0$. Moreover by (A.1) $\psi_1^V(x)$ holds (since we assumed $\psi_1(x)$), so $\varphi(y_1, \dots, y_n)^V \leftrightarrow z = 0$. So $\varphi(y_1, \dots, y_n)$ iff $z = 0$ (by ψ_1) iff $\varphi(y_1, \dots, y_n)^V$ (by ψ_1^V). We have just shown

$$\exists V \forall x, y_1, \dots, y_n, z \in V((\psi_1(x) \wedge x = \langle y_1, \dots, y_n, z\rangle) \to (\varphi(y_1, \dots, y_n) \leftrightarrow \varphi(y_1, \dots, y_n)^V))$$

Moreover under the conditions above $\psi_1(x) \to \psi_3^V$, i.e. $\psi_1(x) \to a \in V$. So

$$\exists V \forall x, y_1, \ldots, y_n, z \in V((\psi_1(x) \wedge x = \langle y_1, \ldots, y_n, z \rangle) \\ \to (a \in V \wedge (\varphi(y_1, \ldots, y_n) \leftrightarrow \varphi(y_1, \ldots, y_n)^V)))$$

Using the standard equivalence $\forall x(\chi(x) \to \rho) \leftrightarrow (\exists x \chi(x) \to \rho)$ (for x, z not free in ρ) we pass to

$$\exists V \forall y_1, \ldots, y_n \in V(\exists x, z \in V(\psi_1'(x, z) \wedge x = \langle y_1, \ldots, y_n, z \rangle) \\ \to (a \in V \wedge (\varphi(y_1, \ldots, y_n) \leftrightarrow \varphi(y_1, \ldots, y_n)^V)))$$

where $\psi_1'(x, z)$ is ψ_1 without quantification over z. It remains to show that

$$\begin{aligned} & \exists x, z \in V(\psi_1'(x, z) \wedge x = \langle y_1, \ldots, y_n, z \rangle) \\ \Leftrightarrow \quad & \exists z \in V(\psi_1'(\langle y_1, \ldots, y_n, z \rangle, z) \wedge \langle y_1, \ldots, y_n, z \rangle \in V \end{aligned}$$

By (A.1) we have

$$(z \in V \wedge \psi_1'(\langle y_1, \ldots, y_n, z \rangle, z)) \to \psi_2^V$$

i.e. implies $\langle y_1, \ldots, y_n, z \rangle) \in V$. So we need to show $\exists z \in V \psi_1'(\langle y_1, \ldots, y_n, z \rangle, z)$.

$\psi_1(x)$ is true iff x is a tuple such that its last value z is chosen to match the "value" of $\varphi(y_1, \ldots, y_n)$. As long as V contains two distinct elements, we can always make such a choice of z in V (choose $z = 0$ if $\varphi(y_1, \ldots, y_n)$ holds, $z = 1 = \{\varnothing\}$ if not. By Accumulation $0 = \varnothing \in V'$ for all V'. Further every stage after the stage $V = \{\varnothing\}$ contains $1 = \{\varnothing\}$). If V does not contain two elements, then $\exists x \in V(x = \langle y_1, \ldots, y_n, z \rangle$ is false as a tuple of $n + 1 \geqslant 2$ elements cannot be the empty set. So $\exists x \in V \psi_1(x)$ is vacuously true and $\exists x, z \in V \psi_1'(x, z)$ is vacuously true as well.

Thus $\exists x, z \in V(\psi_1'(x, z) \wedge x = \langle y_1, \ldots, y_n, z \rangle)$ holds and so the result follows:

$$\exists V \forall y_1, \ldots, y_n \in V(a \in V \wedge (\varphi(y_1, \ldots, y_n) \leftrightarrow \varphi(y_1, \ldots, y_n)^V)) \qquad \square$$